The problem of optimal control with reflection studied through a linear optimization problem stated on occupational measures

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To obtain a linear programming characterization for the minimum cost associated to

$$\begin{cases} i)\ x'(t) \in f(x(t),u(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0 \\ ii)\ x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \text{ and} \\ \ u(\cdot) : [0,\infty) \to U \text{ is a measurable function.} \end{cases} \tag{1}$$

Here K is a nonempty closed subset of $\mathbb{R}^N$, U is a compact metric space, f is a function from $\mathbb{R}^N \times U$ into $\mathbb{R}^N$ and $N_K(x)$ is the normal cone to K at $x \in K$.

The value functions we consider here are given by:

$$V^1(x_0) = \inf_{u(\cdot)\in\mathcal{U}(0)} \int_0^\infty e^{-at} g\left(x\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt \text{ for all } x_0 \in K \quad (2)$$

$$V^2(t_0, x_0) = \inf_{u(\cdot)\in\mathcal{U}(t_0)} g(x(T; t_0, x_0, u(\cdot)) \text{ for all } (t_0, x_0) \in [0, T] \times K \quad (3)$$

where $x(\cdot; t_0, x_0, u(\cdot))$, denotes the solution of (1) starting from (t_0, x_0).

More precisely, we obtain a characterization of the value functions of the following form:

$$\inf_{\gamma \in W^1(x_0)} \int_{K \times U} g(x,u) d\gamma = a V^1(x_0)$$

and respectively

$$\inf_{\gamma \in W^2(t_0,x_0)} \int_{[t_0,T] \times K \times U} g(x) 1_{\{T\} \times K \times U} d\gamma = V^2(t_0,x_0)$$

where $W^1(x_0)$, respectively $W^2(t_0,x_0)$ are sets of probability measures on $K \times U$, respectively $[t_0,T] \times K \times U$. This sets contain the set of occupational measures generated by solutions of the reflected controlled system.

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We assume that $f : \mathbb{R}^N \times U \to \mathbb{R}$ is continuous and satisfies:

$$\begin{cases} ||f(x,u) - f(y,u)|| \leq a \, \|x - y\| & \forall x, y \in \mathbb{R}^N, u \in U \\ \text{The set } f(x,U) \text{ is convex.} \end{cases} \tag{4}$$

where $a > 0$ is a constant.

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Definition 1

A closed set $K \subset \mathbb{R}^N$ is called proximal retract if there exists a neighborhood I of K such that the projection $\Pi_K(\cdot)$ is single-valued in I, with $\Pi_K(x) := \{z \in K \mid ||x - z|| = \inf_{y \in K} ||x - y||\}$ for all $x \in \mathbb{R}$.

So, if K is proximal retract we have that:
- There exist $r, c > 0$ such that the application $x \to N_K(x) \cap B(0,r) + cx$ is monotone. Recall that a set valued map $G : K \to \mathbb{R}^N$is monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for all $y_i \in G(x_i)$, $i \in \{1,2\}$. on K.
- K is sleek i.e. the map $x \to T_K(x)$ is lower semicontinuous (l.s.c.);
- $T_K(x) = C_K(x)$, for all $x \in K$. Here $C_K(x)$ denotes the Clarke's tangent cone. Recall taht $C_K(x) = \{v | \lim_{h \to 0^+, K \ni x' \to x} d_K(x' + hv)/h = 0\}$. This tangent cone is always closed and convex. Note that the class of sleek sets is larger then the class of proximal retracts.

- The map $x \to \Pi_K(x)$ is single valued and continuous on a neighborhood I of K (moreover Π_K is Lipschitz on a neighborhood of K;). Moreover, the map $x \to T_K(x)$ is lower semicontinuous (l.s.c.) and equivalently the polar map $x \to N_K(x)$ has closed graph.

-The map $p : \mathbb{R}^N \to \mathbb{R}_+$ is in $C^{1,1}(I)$ where we recall that

$$p(x) := d_K^2(x) \text{ for all } x \in \mathbb{R}^N.$$

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We consider a closed set K, a set valued map $F : \mathbb{R}^N \to \mathbb{R}^N$ and the following differential inclusions:

$$\begin{cases} i)\, x'(t) \in F(x(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0 \\ ii)\, x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \end{cases} \tag{5}$$

$$\begin{cases} i)\, x'(t) \in \Pi_{\overline{co}T_K(x)} F(x(t)) \text{for almost all } t \geq t_0 \\ ii)\, x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \end{cases} \tag{6}$$

Proposition 2

i) Suppose that K is bounded and F is a set valued map. Then the sets of absolutely continuous solutions to (6) and (5) coincide.
Moreover if F is upper semicontinuous (u.s.c.) with non-empty compact convex values, has a linear growth and K is bounded and sleek, then:
ii) for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^N$ there exists a solution of (5).
iii) the restriction of the map $(t_0, x_0) \to S_{F-N_K}(t_0, x_0)$ to a compact set C of $[0, \infty) \times K$ is compact into $[0, \infty) \times K \times W^{1,1}(0, \infty\,; K)e^{-bt}$ for all b with $b > a_1$. Here $S_{F-N_K}(t_0, x_0)$ denotes the set of solutions to (5) starting from (t_0, x_0).

$$F(x) = f(x,U) = \{f(x,u), u \in U\} \text{ for all } x \in \mathbb{R}^N,$$

For $M > 0$ determined by the maximum of f on , we denote by $S_{f-N_K}(t_0,x_0)$ the set of absolutely continuous solutions to:

$$\begin{cases} i)\ x'(t) \in f(x(t),u(t)) - N_K(x(t)) \cap B(0,M) \text{ for almost all } t \geq t_0 \\ ii)\ x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \\ iii)\ u(\cdot) \in \mathcal{U}(t_0) \end{cases} \tag{7}$$

and by $S_{F-\hat{N}_K}(t_0,x_0)$ the set of absolutely continuous solutions to:

$$\begin{cases} i)\ x'(t) \in F(x(t)) - N_K(x(t)) \cap B(0,M) \text{ for almost all } t \geq t_0 \\ ii)\ x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0. \end{cases} \tag{8}$$

Proposition 3

Suppose that K is a compact proximal retract set and (H_f) holds.
i) If $x(\cdot)$ is a solution to (8) starting from (t_0, x_0) then there exists $u(\cdot) \in \mathcal{U}(t_0)$ such that $x(\cdot)$ is equal to $x(\cdot; t_0, x_0, u(\cdot))$, solution of (7).
ii) As a direct consequence of i):

$$S_{F-N_K}(t_0, x_0) = S_{f-N_K}(t_0, x_0) \text{ for all } (t_0, x_0) \text{ in } [0, T] \times K.$$

Lemma 4

Assume that (H_f) holds true and K is a bounded proximal retract. Then for $x_0(\cdot) \in S_{f-N_K}$, $x_1(\cdot) \in S_{f-N_K}(t_1, x_1)$ with x_1, x_2 in K, fixed $u(\cdot) \in \mathcal{U}(t_0)$ and $t \geq t_1 \geq t_0$ there exists $C > 0$ a constant depending on t, such that:

$$\|x_0(t; t_0, x_0, u(\cdot)) - x_1(t; t_1, x_1, u(\cdot))\| \leq C(\|x_0 - x_1\| + |t_0 - t_1|).$$

As a direct consequence of the above estimation we obtain:

Corollary 5

Assume that (H_f) holds true and K is a bounded proximal retract. Then for every fixed $u(\cdot) \in \mathcal{U}(t_0)$ there exists an unique solution of (1) in K.

Lemma 6

Suppose that (H_f) holds true and K is a compact proximal retract. Then we have
(Existence of an Optimal control) If g is lower semicontinuous, then V^1 and V^2 are lower semicontinuous and there exists optimal trajectories starting from each point x_0 and respectively (t_0, x_0), i.e. there exists $\bar{x}_1(\cdot), \bar{x}_2(\cdot) \in S_{F-N_K}(t_0, x_0)$ such that

$$V^1(x_0) = \int_0^\infty e^{-at} g\left(\bar{x}_1\left(t; x_0, \bar{u}_1(\cdot)\right), \bar{u}_1(\cdot)\right) dt$$

$$V^2(t_0, x_0) = g(\bar{x}_2(T; t_0, x_0, \bar{u}_2(\cdot))).$$

Lemma 7

(Dynamic Programming Principle) Let $g : K \to \mathbb{R}$ *be a bounded function,* K *a compact proximal retract and suppose that* (H_f) *holds. Then, for all* $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ *and* $h > 0$ *small enough we have:*

$$V^1(x_0) = \inf_{u(\cdot)\in\mathcal{U}^K(0)} \left\{ \int_0^h e^{-at} g\left(x\left(t; x_0, u\right), u\right) dt + V^1(x\left(t+h; x_0, u\right)) \right\},$$

$$V^2(t_0, x_0) = \inf_{x\in S_{F-N_K}(t_0,x_0)} V^2(t_0+h, x(t_0+h)).$$

Outline

We give now a brief idea of our method. It consists in introducing the following approximating control systems:

$$\begin{cases} i)\ x'(t) = f(x(t), u(t)) - n\nabla p(x(t)) \text{ for almost all } t \geq t_0 \\ ii)\ x(t_0) = x_0 \in K_n \\ \ u(\cdot) : [0, \infty) \to U \text{ is a measurable function} \end{cases} \tag{9}$$

where $n \in N^*$, K_n will be defined latter. The function $p : \mathbb{R}^N \to \mathbb{R}_+$ is defined by

$$p(x) := d_K^2(x) \text{ for all } x \in \mathbb{R}^N.$$

Here we note by $d_A(x) := \inf_{y \in A} ||x - y||$ the distance function to a set $A \subset \mathbb{R}^N$. In this paper $|| \cdot ||$ and $\langle \cdot \rangle$ are the Euclidian norm and scalar product in $\mathbb{R}^N$. Moreover, B denotes the closed unit sphere of $\mathbb{R}^N$. Under appropriate hypotheses on K, the function p will be in $C^{1,1}(K_n)$.

The value functions associated with (9) are given by:

$$V_n^1(x_0) = \inf_{u(\cdot)\in\mathcal{U}(0)} \int_0^{\infty} e^{-at} g\left(x_n\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt \text{ for all } x_0 \in K_n \tag{10}$$

$$V_n^2(t_0, x_0) = \inf_{u(\cdot)\in\mathcal{U}(t_0)} g(x_n(T; t_0, x_0, u(\cdot)) \text{ for all } (t_0, x_0) \in [0, T] \times K_n. \tag{11}$$

Here $x_n(\cdot; t_0, x_0, u(\cdot))$ denotes the solution of (9) starting from (t_0, x_0) and $\mathcal{U}(t_0)$ is the set of measurable controls on $[t_0, \infty)$ with values in U.

Proposition 8

Suppose that K is a compact proximal retract set and (4) holds. Then for n large enough the set $K_n := K + \frac{M}{n}B$ is contained in I and it is invariant for (9).

Proposition 9

Suppose that K is a compact proximal retract set and (4) holds. Then for every $u(\cdot) : [0, \infty) \to U$ a measurable control the sequence of solutions $x_n(\cdot, t_0, x_0, u(\cdot))$

$$\begin{cases} i)\ x'(t) = f(x(t), u(t)) - n\nabla p(x(t)) \text{ for almost all } t \geq t_0 \\ ii)\ x(t_0) = x_0 \in K_n \end{cases} \tag{12}$$

contains a subsequence which converges (weekly) to the solution of

$$\begin{cases} i)\ x'(t) \in f(x(t), u(t)) - N_K(x(t)) \cap B(0, M) \text{ for almost all } t \geq t_0 \\ ii)\ x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \end{cases} \tag{13}$$

Conversely, we have that any solution of (1) can be approximated (weekly) by a sequence of solutions of (9).

Now, we can easily prove that

Proposition 10

Suppose that K is a compact proximal retract set and (H_f) holds. Then for $i \in \{1,2\}$

$$V_n^i \to V^i \text{ pointwisely when } n \to \infty.$$

Outline

Note that if $g : \mathbb{R}^N \to \mathbb{R}$ is a continuous function then the value functions are also continuous.

$$aV_n^1(x_0) = \inf_{u \in \mathcal{U}} \int_{K_n \times U} g(x,u) d\gamma_{(n,u)}$$

Here for all $x_n(\cdot; x_0, u(\cdot))$ we associate the probability measure $\gamma_{(n,u)} \in P(K_n \times U)$ given by

$$\gamma_{(n,u)}(Q) := a \int_0^\infty e^{-at} 1_Q \left(x_n\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt$$

for any $Q \subset K_n \times U$ which is borelian. The set of all this discounted occupational measures is denoted by $\Gamma_n^1(x_0)$. Equivalently, the previous definition can be expressed by

$$\int_{K_n \times U} l(x,u) d\gamma_{(n,u)} = a \int_0^\infty e^{-at} l\left(x\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt.$$

for any continuous function $l : K_n \times U \to \mathbb{R}$.

Definition 11

For every $x_0 \in K_n$ we denote by $Y_n := K_n \times U$

$$W_n^1(x_0) := \left\{ \begin{array}{c} \gamma \in P(Y_n) \mid \forall \varphi \in C^1(K_n;\mathbb{R}), \\ \int_{Y_n} [\langle \nabla\varphi(x), f(x,u) - n\nabla p(x)\rangle - a(\varphi(x_0) - \varphi(x))]\, d\gamma = 0. \end{array} \right\}$$

$$\Phi_n^1(x_0) := \left\{ \begin{array}{c} \varphi \in C^1(K_n;\mathbb{R}) \text{ such that} \\ -a\varphi(x) + \langle \nabla\varphi(x), f(x,u) - n\nabla p(x)\rangle + g(x,u) \geq 0 \\ \text{for all } (x,u) \in K_n \times U \end{array} \right\}$$

$$\mu_n^1(x_0) := \sup \left\{ \begin{array}{c} \mu_n \in R \mid \exists \varphi \in C^1(K_n;\mathbb{R}) \text{ such that} \\ \mu \leq \langle \nabla\varphi(x), f(x,u) - n\nabla p(x)\rangle - a(\varphi(x_0) - \varphi(x)) \\ +g(x,u) \text{ for all } (x,u) \in K_n \times U \end{array} \right\}$$

We obtain the equality

$$\inf_{\gamma \in W_n^1(x_0)} \int_{K_n \times U} g(x,u) d\gamma_{(n,u)} = \mu_n^1(x_0) = aV_n^1(x_0).$$

Similar result holds for V_n^2.

Proposition 12

Suppose that K is a compact proximal retract and (4) holds. If g is l.s.c. then for all x_0 in $K_{\frac{n}{2}}$ we have:

$$V_n^1(x_0) = \sup\{\varphi(x_0) | \varphi \in \Phi_\infty^n \text{ and } \varphi(\cdot) \leq g(\cdot) \text{ on } K_n\}$$
$$V_n^2(t_0, x_0) = \sup\{\varphi(t_0, x_0) | \varphi \in \Phi_T^n \text{ and } \varphi(T, \cdot) \leq g(\cdot) \text{ on } K_n\}$$

Here Φ_∞^n is the set of all functions $\varphi : \mathbb{R}^N \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^N; \mathbb{R})$ such that

$$-a\varphi(x) + \langle \nabla\varphi(t,x), f(x,u) - n\nabla p(x)\rangle + g(x,u) \geq 0 \quad \text{for all } (x,u) \in K_n \times U$$

and Φ_T^n is the set of all functions $\varphi : \mathbb{R}^{N+1} \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^{N+1}; \mathbb{R})$ such that

$$\nabla_t\varphi(t,x) + \langle \nabla_x\varphi(t,x), f(x,u) - n\nabla p(x)\rangle \geq 0$$

Note that that if g is continuous then

- V^1 is viscosity solution of the following Hamilton Jacobi Inclusion

$$-aV(x) + H_1(x, \nabla V(x)) - \langle \nabla V(x), N_K(x) \rangle \ni 0 \text{ if } x \in K \tag{14}$$

- V^2 is viscosity solution of the following Hamilton Jacobi Inclusion

$$\begin{cases} \nabla_t V(t,x) + H_2(x, \nabla_x V(t,x)) - \langle \nabla_x V(t,x), N_K(x) \rangle \ni 0 \\ \text{if } (t,x) \in [0,T) \times \mathbb{K}; \\ \text{with the condition } V(T,x) = g(x) \text{ if } x \in K. \end{cases} \tag{15}$$

Moreover using classical results we have

- V_n^1 is viscosity solution of the following Hamilton Jacobi Inclusion

$$-aV(x) + H_1(x, \nabla V(x)) - \langle \nabla V(x), n\nabla p(x) \rangle = 0 \text{ if } x \in K \tag{16}$$

- V_n^2 is viscosity solution of the following Hamilton Jacobi Equation

$$\begin{cases} \nabla_t V(t,x) + H_2(x, \nabla_x V(t,x)) - \langle \nabla V_x(t,x), n\nabla p(x) \rangle = 0 \\ \text{if } (t,x) \in [0,T) \times \mathbb{K}; \\ \text{with the condition } V(T,x) = g(x) \text{ if } x \in K. \end{cases} \tag{17}$$

Here the Hamiltonians are given by:

$$H_1(x,p) = \min_{u\in U}(\langle f(x,u),p\rangle + g(x,u)), \forall (x,p) \in K \times \mathbb{R}^N,$$
$$H_2(x,p) = \min_{u\in U}\langle f(x,u),p\rangle, \forall (x,p) \in K \times \mathbb{R}^N.$$

For the convenience of the reader we recall the notion of solution that we employ here (for instance for (15)).

Definition 13

A *viscosity supersolution of (*14) is a lower semicontinuous (l.s.c. in short) function $\psi : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ such that:

$$\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg Min\,(\psi - \phi)\,,$$
$$\text{there exists } y_0 \in N_K(x_0) \text{ such that}$$
$$\nabla_t\phi(t_0,x_0) + H_2(x_0, \nabla_x\phi(t_0,x_0)) - \langle y_0, \nabla_x\phi(t_0,x_0)\rangle \leq 0.$$

and a *viscosity subsolution of (14)* is an upper semicontinuous (u.s.c. in short) function $\varphi : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ such that:

$$\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg Max\,(\varphi - \phi)\,,$$
$$\text{there exists } z_0 \in N_K(x_0) \text{ such that}$$
$$\nabla_t\phi(t_0,x_0) + H_2(x_0, \nabla_x\phi(t_0,x_0)) - \langle z_0, \nabla_x\phi(t_0,x_0)\rangle \geq 0.$$

Outline

Note that we have

$$aV^1(x_0) = \inf_{u \in \mathcal{U}} \int_{K \times U} g(x,u) d\gamma_{(u)}$$

As previously, for all $x(\cdot; x_0, u(\cdot))$ we associate the probability measure $\gamma_{(u)} \in P(K \times U)$ given by

$$\gamma_{(u)}(Q) := a \int_0^\infty e^{-at} 1_Q \left(x\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt.$$

for any $Q \subset K \times U$ which is borelian. Equivalently, the previous definition can be expressed by

$$\int_{K \times U} l(x,u) d\gamma_{(u)} = a \int_0^\infty e^{-at} l \left(x\left(t; x_0, u(\cdot)\right), u(\cdot)\right) dt.$$

for any continuous function $l : K \times U \to \mathbb{R}$.
The set of all measures associated with trajectories is denoted by $\Gamma^1(x_0)$.
We denote by $\gamma_n \rightharpoonup \gamma$ where γ_n, γ are probability measures on $K \times U$ the weak convergence i.e.

Definition 14

For every $x_0 \in K$ we denote by $Y := K \times U$

$$W^1(x_0) := \{ \ \gamma \in P(Y) \mid \exists \gamma_n \in W_n^1(x_0) \text{ such that } \gamma_n \rightharpoonup \gamma \ \}$$

$$\Phi_\infty(x_0) := \cup_{n \in N} \Phi_n^1(x_0)$$

$$\mu^1(x_0) := \sup \left\{ \begin{array}{c} \mu \in \mathbb{R} \mid \exists \varphi \in C^1(\mathbb{R}^N; \mathbb{R}) \text{ and } n \in N \\ \mu \leq \langle \nabla \varphi(x), f(x,u) - n \nabla p(x) \rangle - a(\varphi(x_0) - \varphi(x)) \\ +g(x,u) \text{ for all } (x,u) \in K_n \times U \end{array} \right\}$$

$$:= \sup_{n \in N} \mu_n^1(x_0).$$

Theorem 15

Suppose that K is a compact proximal retract set and (4) holds. Then, for all $x_0 \in K$ we have that

$$aV^1(x_0) = \inf_{\gamma \in W^1(x_0)} \int_{K \times U} g(x,u)d\gamma = \mu^1(x_0).$$

Outline

Suppose that for $x_0 \in K$ we have the following property ; there exists a sequence

$$\varphi_{n_k} \in \Phi^1_{n_k}(x_0) \text{ such that } \lim_k \varphi_{n_k}(x_0) \to V^1(x_0) \text{ and } \langle \nabla \varphi_{n_k}, \nabla p \rangle \leq 0 \text{ on } K_{n_k}. \tag{18}$$

For every $x_0 \in K$ we denote by $Y := K \times U$

$$\tilde{W}^1(x_0) := \left\{ \begin{array}{c} \gamma \in P(Y) \mid \forall \varphi \in C^1(\mathbb{R}^N;\mathbb{R}),\ \exists y(\cdot) \text{ selection of } N_K(\cdot) \\ \int_Y \left[\left\langle \frac{\partial \phi}{\partial x}(x), f(x,u) - y(x) \right\rangle - a(\varphi(x_0) - \varphi(x)) \right] d\gamma = 0. \end{array} \right\}$$

$$\tilde{\Phi}_\infty(x_0) := \left\{ \begin{array}{c} \varphi \in C^1(K;\mathbb{R}) \text{ such that } \forall y(\cdot) \text{ selection of } N_K(\cdot) \cap B(0,M) \\ -a\varphi(x) + \left\langle \frac{\partial \phi}{\partial x}(x), f(x,u) - y(x) \right\rangle + g(x,u) \geq 0 \\ \text{for all } (x,u) \in K \times U \end{array} \right\}$$

$$\tilde{\mu}^1(x_0) := \sup \left\{ \begin{array}{c} \mu \in R \mid \exists \varphi \in C^1(\mathbb{R}^N;\mathbb{R}) \text{ such that } \forall y(\cdot) \text{ selection of } N_K(\cdot) \cap B \\ \mu \leq \left\langle \frac{\partial \phi}{\partial x}(x), f(x,u) - y(x) \right\rangle - (a\varphi(x_0) - \varphi(x)) \\ +g(x,u) \text{ for all } (x,u) \in K \times U \end{array} \right.$$

Theorem 16

Suppose that K is a compact proximal retract set and (4) holds. Then, for all $x_0 \in R^N$ we have that

$$aV^1(x_0) = \inf_{\gamma \in \tilde{W}^1(x_0)} \int_Y g(x,u)d\gamma = \tilde{\mu}^1(x_0).$$

Outline

Note that if $g : \mathbb{R}^N \to \mathbb{R}$ then we have the following

$$V_n^2(t_0, x_0) = \inf_{u \in \mathcal{U}} \int_{[t_0,T]\times K_n \times U} 1_{\{T\}\times K \times U} g(x) d\gamma_{(n,u)}$$

Here for all $x_n(\cdot; x_0, u(\cdot))$ we associate the probability measure $\gamma_{(n,u)} \in P([t_0, T] \times K_n \times U)$ given by

$$\gamma_{(n,u)}(Q) := \frac{1}{T - t_0} \int_{t_0}^{T} 1_Q(t, x(t), u(t))dt$$

for any Q which is measurable subset of $[t_0, T] \times K_n \times U$. Equivalently, the previous definition can be expressed by

$$\int_{K_n \times U} l(x, u)d\gamma_{(n,u)} = \frac{1}{T - t_0} \int_{t_0}^{T} l(t, x(t), u(t))dt.$$

for any continuous function $l : [t_0, T] \times K_n \times U \to \mathbb{R}$.

For every $(t_0, x_0) \in [t_0, T] \times K_n$ we denote by $Y_n := K_n \times U$

$$W^2_{n,T}(t_0,x_0) := \left\{ \begin{array}{c} \gamma \in P([t_0,T]\times Y_n) \mid \forall \varphi \in C^1([t_0,T]\times K_n;\mathbb{R}), \\ \int_{[t_0,T]\times K_n \times U} \left[(T-t_0)\left(\nabla_t \varphi(t,x) + \langle \nabla_x \varphi(t,x), f(x,u) - n\nabla p(x \right.\right. \\ \left. -1_{\{T\}\times \mathbb{K}\times U}\varphi(s,x) + \varphi(t_0,x_0)\right] d\gamma = 0 \end{array} \right.$$

$$\Phi^2_n(t_0,x_0) := \left\{ \begin{array}{c} \varphi \in C^1([t_0,T]\times K_n;\mathbb{R}) \text{ such that} \\ (T-t_0)\left(\nabla_t \varphi(s,x) + \langle \nabla_x \varphi(s,x), f(x,u) - n\nabla p(x)\rangle\right) \\ -1_{\{T\}\times \mathbb{K}\times U}\varphi(s,x) + \varphi(t_0,x_0) \geq 0 \\ \text{for all } (t,x,u) \in [t_0,T]\times K_n \times U \end{array} \right\}$$

$$\mu^2_n(t_0,x_0) := \sup \left\{ \begin{array}{c} \mu_n \in R \mid \exists \varphi \in C^1([t_0,T]\times K_n;\mathbb{R}) \text{ such that} \\ \mu \leq (T-t_0)\left(\nabla_t \varphi(t,x) + \langle \nabla_x \varphi(t,x), f(x,u) - n\nabla p(x)\rangle\right) \\ -1_{\{T\}\times K_n \times U} g(x) + \varphi(t_0,x_0) \\ \text{for all } (t,x,u) \in [t_0,T]\times K_n \times U \end{array} \right\}$$

Definition 17

For every $(t_0, x_0) \in [t_0, T] \times K$ we denote by $Y := K \times U$

$$W^2(t_0, x_0) := \left\{ \begin{array}{c} \gamma \in P([t_0, T] \times Y) \mid \exists \gamma_n \in W_n^2(t_0, x_0) \text{ such that} \\ \gamma_n \rightharpoonup \gamma \end{array} \right\}$$

$$\Phi_\infty^2(t_0, x_0) := \cup_{n \in N} \Phi_n^2(t_0, x_0)$$

$$\mu^2(t_0, x_0) := \sup_{n \in N} \mu_n^2(t_0, x_0).$$

Theorem 18

Suppose that K is a compact proximal retract set and (4) holds. Then, for all (t_0, x_0)*:*

$$V^2(t_0, x_0) = \inf_{\gamma \in W^2(t_0, x_0)} \int_{[t_0, T] \times K \times U} g(x) 1_{\{T\} \times K \times U} d\gamma = \mu^2(t_0, x_0).$$

Outline

Consider the solution $x(\cdot; x_0, u(\cdot))$ of (1), i.e. $\exists y(\cdot)$ selection of $N_K(\cdot)$ such that

$$x'(t) = f(x(t), u(t)) - y(x(t)) \; a.e.$$

For any $\varphi \in C^1(\mathbb{R}^N; \mathbb{R})$ it is easy to see that

$$\int_0^\infty \frac{d}{dt} e^{-at} \varphi(x(t)) dt = -\varphi(x_0)$$

So,

$$\int_0^\infty e^{-at} \left(-a\varphi(x(t)) + \left\langle \frac{\partial \phi}{\partial x}(x(t)), f(x(t), u(t)) - y(x(t)) \right\rangle \right) dt = -\varphi(x_0)$$

or equivalently

$$\int_Y \left(-a\varphi(x) + \left\langle \frac{\partial \phi}{\partial x}(x), f(x(t), u(t)) - y(x(t)) \right\rangle \right) d\gamma_{(u)} = -a\varphi(x_0)$$

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