Commande optimale d'équations elliptiques semi linéaires par pénalisation interne

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For $s > \frac{1}{2}n$ consider the following problem

$$(\mathcal{CP}_0^s) : \begin{cases} \mathrm{Min}\ \frac{1}{2}\int_\Omega (y(x) - \bar{y}(x))^2 dx + \frac{1}{2}N\int_\Omega u(x)^2 \mathrm{d}x \\ \text{s.t} \quad \begin{array}{rcll} -\Delta y(x) + \phi(y(x)) & = & f(x) + u(x) & \text{for} \quad x \in \Omega. \\ y(x) & = & 0 & \text{for} \quad x \in \partial\Omega \\ u(x) & \in & \mathcal{U}_+^s. & \end{array} \end{cases}$$

Here ϕ is $\mathcal{C}^2$, increasing and Lipschitz with associated constant L. The boundary of Ω is $\mathcal{C}^2$, $\bar{y} \in \mathcal{C}^2$, $N > 0$ and $\mathcal{U}_+^s := L^s(\Omega; \mathbb{R}_+)$.

- For every $u \in L^s$, the semilinear elliptic equation admits a unique solution noted $y_u \in \mathcal{Y}^s := W^{2,s} \cap W_0^{1,s}$.
- Define

$$J_0(u) := \frac{1}{2}\int_\Omega (y_u(x) - \bar{y}(x))^2 dx + \frac{1}{2}N\int_\Omega u(x)^2 \mathrm{d}x$$

- Note that $J_0(u)$ is not necessarily convex, therefore the classical proof of existence and uniqueness does not apply.

Instead we have

Proposition

Problem $(\mathcal{CP}_0^s)$ has (at least) one solution u_0.

Now, for every $u \in L^s$ let us define the adjoint state p_u associated with u as the unique solution in $\mathcal{Y}^s$ of

$$\begin{cases} -\Delta p_u(x) + \phi'(y_u(x))p_u(x) = y_u(x) - \bar{y}(x) & \text{for} \quad x \in \Omega \\ p_u(x) = 0 & \text{for} \quad x \in \partial\Omega \end{cases}$$

We will write $y_0 := y_{u_0}$ and $p_0 := p_{u_0}$. F.O.C. for $(\mathcal{CP}_0^s)$ imply that

$$u_0(x) = \Pi_0(-N^{-1}p_0(x)) \quad \text{for a.a } x \in \Omega$$

where $\Pi_0(x) := \max\{x, 0\}$.

Next we consider a *localized* penalized version of $(\mathcal{CP}_0^s)$. Let ℓ be a convex $\mathcal{C}^2$ function such that

- $$\lim_{x\to 0^+} \ell'(x) = -\infty$$
- $$\lim_{x\to 0^+} \frac{\ell''(x)}{\ell'(x)} = -\infty.$$
- There exist constants $\alpha_1, \alpha_2 \geq 0$ such that

$$|\ell'(t)| \leq \alpha_1 t + \alpha_2 \quad \forall\ t \geq 1.$$

Examples:

- $\ell(x) = -\log x$ (logarithmic barrier)
- $\ell(x) = \dfrac{1}{x^p} \quad p\ >\ 0, \qquad \ell(x) = x \log x.$

Penalized problem

Let u_0 be a solution of $(\mathcal{CP}_0^s)$. For $b, \varepsilon > 0$ consider the problem $(\mathcal{CP}_\varepsilon^{b,s})$ defined as

$$\mathrm{Min}\ \ J_\varepsilon(u) := J_0(u) + \varepsilon \int_\Omega \ell(u(x))\mathrm{d}x \quad \text{s. t. } u \in \mathcal{U}_+^s \cap \bar{B}_s(u_0, b).$$

where $\bar{B}_s(u_0, b)$ denotes the closed ball in L^s centered at u_0 of radius b. It holds that

Proposition

Problem $(\mathcal{CP}_\varepsilon^{b,s})$ has (at least) a solution u_ε. In addition, there exists a constant $C_1 > 0$ such that

$$\ell'(2u_\varepsilon(x)) \geq -\frac{2C_1}{\varepsilon} \quad \textit{for a.a. } x \in \Omega.$$

and, if ε is small enough, there exists a constant K_ℓ such that

$$u_\varepsilon(x) \leq K_\ell \quad \textit{for a.a. } x \in \Omega.$$

Note that $u \in L^s \to \int_\Omega \ell(u(x))\mathrm{d}x$ is, in general, not continuous and therefore not differentiable. Nevertheless, thanks to the proposition above we have that $u_\varepsilon \in L^\infty$ which allows us to write some first order optimality conditions.

Lemma

Let u_ε be a local solution of $(\mathcal{CP}_\varepsilon^{b,s})$. Then there exist $\lambda_\varepsilon \geq 0$ such that

$$\begin{aligned} Nu_\varepsilon(x) + p_\varepsilon(x) + \varepsilon\ell'(u_\varepsilon) + \lambda_\varepsilon u_\varepsilon(x)^{s-1} &= 0 \quad \text{for a.a. } x \in \Omega \\ \lambda_\varepsilon\left(||u_\varepsilon - u_0||_s - b\right) &= 0. \end{aligned}$$

Now we can state the convergence result

Proposition

Suppose that there exists $b_0 > 0$ such that u_0 is a strict local minimum of $(\mathcal{CP}_0^s)$ in $B_s(u_0, b_0)$. Then,
(i) *The controls u_ε, solutions of $(\mathcal{CP}_\varepsilon^{b,s})$ converge as $\varepsilon \downarrow 0$ to u_0 in L^s.*
(ii) *It holds that $J_\varepsilon(u_\varepsilon) \longrightarrow J_0(u_0)$ and that $J_0(u_\varepsilon) \downarrow J_0(u_0)$.*
(iii) *The states y_ε converge to y_0 in $W^{2,s}$ and the adjoint states p_ε converge to p_0 in $W^{2,s}$.*

Convergence results and optimality conditions imply that $\lambda_\varepsilon = 0$ for ε small enough and therefore

$$u_\varepsilon(x) = \Pi_\varepsilon(-N^{-1} p_\varepsilon(x))$$

where $\Pi_\varepsilon(x)$ is defined as the solution of

$$\operatorname*{Min}_{z \in \mathbb{R}_+} \ \tfrac{1}{2}(z - x)^2 + \varepsilon \ell(z),$$

Asymptotic expansion and error estimation

Let u_0 be a solution of $(\mathcal{CP}_0^s)$ and y_0, p_0 its associated state and costate, respectevely. Consider the mapping $F : \mathcal{Y}^s \times \mathcal{Y}^s \times \mathbb{R}_+ \to L^s \times L^s$ defined by

$$F(y, p, \varepsilon) := \begin{pmatrix} \Delta y + \Pi_\varepsilon(-N^{-1}p) + f - \phi(y) \\ \Delta p + y - \bar{y} - \phi'(y)p \end{pmatrix}.$$

The objective is to obtain an "asymptotic expansion" for $(y_\varepsilon, p_\varepsilon)$, the state and costate of a *localized penalized* problem, around (y_0, p_0).

• It is easy to see that in general F is not differentiable at $(y_0, p_0, 0)$. Therefore, we cannot apply the standard implicit function theorem in order to obtain such expansion.

Theorem

(Restoration theorem) *Let X and Y be Banach spaces, E a metric space and $F : U \subset X \times E \to Y$ a continuous mapping on an open set U. Let $(\hat{x}, \varepsilon_0) \in U$ be such that $F(\hat{x}, \varepsilon_0) = 0$. Assume that there exists a surjective linear continuous mapping $A : X \to Y$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(\beta) \downarrow 0$ when $\beta \downarrow 0$ such that, if $x \in \overline{B}(\hat{x}, \beta)$, $x' \in \overline{B}(\hat{x}, \beta)$ and $\varepsilon \in B(\varepsilon_0, \beta)$, then*

$$\|F(x', \varepsilon) - F(x, \varepsilon) - A(x' - x)\| \leq c(\beta)\|x' - x\|. \qquad (1)$$

Then, denoting by B a bounded right inverse of A, for ε close to ε_0, $F(\cdot, \varepsilon)$ has, in a neighborhood of $\hat{x}$, a zero denoted by x_ε such that the following expansion holds

$$x_\varepsilon = \hat{x} - BF(\hat{x}, \varepsilon) + r(\varepsilon) \quad \text{with } ||r(\varepsilon)|| = o\left(\|F(\hat{x}, \varepsilon)\|\right). \qquad (2)$$

Since $(\mathcal{CP}_0^s)$ is not convex (in general) we will impose a second order condition at u_0.
Let us consider a more general framework. Let $K \subseteq L^2$ be a nonempty closed and convex set and define $K_s := K \cap L^s$.
Consider the problem

$$\text{Min } J_0(u) \text{ subject to } u \in K_s \qquad (\mathcal{AP})$$

For $u \in K_s$ define

$$\begin{aligned} C(u) &:= \{v \in L^2 \,;\, v \in T_K(u) \text{ and } DJ_0(u)v \leq 0\} \\ C_s(u) &:= \{v \in L^s \,;\, v \in T_{K_s}(u) \text{ and } DJ_0(u)v \leq 0\}. \end{aligned}$$

where $T_K(u)$ denotes the *tangent cone* to K at u in L^2 and $T_{K_s}(u)$ is the *tangent cone* to K_s at u in L^s.

• The set K_s is polyhedric in L^s at $u \in K_s$ if, for all $u^* \in N_{K_s}(u)$ (sets of normal of K_s at u), the set $\mathcal{R}_{K_s}(u) \cap (u^*)^{\perp}$ is dense in $T_{K_s}(u) \cap (u^*)^{\perp}$. If K_s is polyhedric in L^s at each $u \in K_s$ we say that K_s is s-polyhedric.

• We say that J_0 satisfies the local quadratic growth condition at $u \in K_s$ if there exists $\alpha > 0$ and a neighborhood $\mathcal{V}_s \subseteq L^s$ of u such that

$$J_0(u') \geq J_0(u) + \alpha ||u' - u||_2^2 + o(||u' - u||_2^2) \quad \text{for all } u' \in K_s \cap \mathcal{V}_s.$$

It holds that

Theorem

Suppose that $u \in K_s$. If K_s is s-polyhedric and $C_s(u)$ is dense in $C(u)$, then the quadratic growth condition, the second order condition

$$\exists \alpha > 0, \text{such that } D^2 J_0(u)(v,v) \geq \alpha ||v||_2^2 \quad \text{for all } v \in C(u)$$

and the punctual relation

$$D^2 J_0(u)(v,v) > 0 \quad \text{for all } v \in C(u) \setminus \{0\}$$

are equivalent.

We have that

Lemma

If $K_s = \mathcal{U}^s_+$, then K_s is s-polyhedric and $\overline{C_s(u)}^{L^2} = C(u)$.

We will assume the following hypothesis.

(H1) For the adjoint state p_0, asociated to the solution u_0 of $(\mathcal{CP}^s_0)$, it holds that

$$\text{mes}(\{x \in \Omega\ ;\ p_0(x) = 0\}) = 0.$$

(H2) At any local solution u_0 of $(\mathcal{CP}^s_0)$, it holds that

$$\exists \alpha > 0, \text{such that } D^2 J_0(u)(v,v) \geq \alpha ||v||_2^2 \quad \text{for all } v \in C(u).$$

For $h \in L^2$ let z_h be the unique solution of

$$-\Delta z + \phi'(y_0) z = h \quad \text{in } \Omega \; ; \qquad z = 0 \quad \text{in } \partial\Omega$$

Lemma

If assumptions **(H1)** *and* **(H2)** *hold, then F is differentiable with respect to (y,p) at $(y_0, p_0, 0)$ and the linear application $D_{(y,p)}F(y_0,p_0,0)$ is an isomorphism.*
In addition, for every $(\delta_1, \delta_2) \in L^s \times L^s$, we have that

$$D_{(y,p)}F(y_0,p_0,0)^{-1}(\delta_1,\delta_2)$$

is the unique solution of the reduced optimality system of

$$Min \int_\Omega \left[\tfrac{1}{2} N v^2 + \tfrac{1}{2}\left(1 - p_0 \phi''(y_0)\right) z_{v+\delta_1}^2 + \delta_2 z_{v+\delta_1}\right] \mathrm{d}x \qquad (\mathcal{QP}_{\delta_1,\delta_2})$$

subjet to $v \in C(u_0)$.

Define $q_0 := -p_0/N$.

Theorem

Suppose that **(H1)** *and* **(H2)** *hold and let* $u_0 \in \mathcal{U}^s$ *be any local solution of* $(\mathcal{CP}_0^s)$*. Denote by* y_0 *and* p_0 *its associated state and adjoint state respectively. Then there are* $\bar{b} > 0$ *and* $\bar{\varepsilon} > 0$ *such that for* $\varepsilon \in [0, \bar{\varepsilon}]$ *problem* $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ *has a unique solution* u_ε*. In addition, denoting by* y_ε *and* p_ε *the associated state and adjoint state for* u_ε*, the following expansion around* (y_0, p_0) *holds*

$$\begin{pmatrix} y_\varepsilon \\ p_\varepsilon \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix} + D_{(y,p)}F(y_0, p_0, 0)^{-1} F(y_0, p_0, \varepsilon) + r(\varepsilon), \quad (3)$$

where $r(\varepsilon) = o(||F(y_0, p_0, \varepsilon)||_s)$*. Moreover,* $D_{(y,p)}F(y_0, p_0, 0)^{-1} F(y_0, p_0, \varepsilon)$ *is characterized as being the unique solution of* $(\mathcal{QP}_{\delta\Pi(\varepsilon),0})$ *where*

$$\delta\Pi(\varepsilon) := \Pi_\varepsilon(q_0) - \Pi_0(q_0).$$

Corollary (Error bounds)

Under the assumptions of theorem 9 we have

(i) *The error estimates for $u_\varepsilon, y_\varepsilon$ and p_ε are given by*

$$||u_\varepsilon - u_0||_s + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(||\delta\Pi(\varepsilon)||_s\right). \quad (4)$$

(ii) *The error bound for the control in the infinity norm is given by*

$$||u_\varepsilon - u_0||_\infty = O\left(||\delta\Pi(\varepsilon)||_\infty\right). \quad (5)$$

(iii) *The error estimate for the cost is given by*

$$|J_0(u_\varepsilon) - J_0(u_0)| = O\left(||\delta\Pi(\varepsilon)||_s\right). \quad (6)$$

Logarithmic barrier

In this section we apply the results obtained to an important example, which is when $\ell(x) = -\log(x)$.

Theorem

Suppose that $\ell(x) = -\log x$ and that hypothesis **(H1)** *and* **(H2)** *hold. Let $\bar{b} > 0$ be such that $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ has a unique solution for $\varepsilon > 0$ small enough. Then*

(i) *We have*

$$||u_\varepsilon - u_0||_\infty + ||p_\varepsilon - p_0||_{2,s} + ||y_\varepsilon - y_0||_{2,s} = O(\sqrt{\varepsilon}), \tag{7}$$

$$|J_0(u_\varepsilon) - J_0(u_0)| = O(\varepsilon). \tag{8}$$

Theorem (Continuation...)

(ii) *If in addition $n \leq 3$, there exists $m \in \mathbb{N}$ such that*

$$\{x \in \Omega \ ; \ p_0(x) = 0\} = \bigcup_{i=1}^{m} C_i \tag{9}$$

where for every $i \in \{1, ..., m\}$, the set C_i is a regular closed curve and

$$\min_{\{x \in \Omega \ ; \ p_0(x)=0\}} \left| \frac{\partial p_0}{\partial \hat{n}}(x) \right| \neq 0, \tag{10}$$

then

$$||u_\varepsilon - u_0||_2 + ||p_\varepsilon - p_0||_{2,2} + ||y_\varepsilon - y_0||_{2,2} = O(\varepsilon^{\frac{3}{4}}). \tag{11}$$

References

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